The Speed of Cooling Fronts and the Functional Form of the Dimensionless Viscosity in Accretion Disks

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ABSTRACT

We examine the speed of inward traveling cooling fronts in accretion disks. We show that their speed is determined by the rarefaction wave that precedes them and is approximately $\alpha_F c_F (H/r)^q$, where α_F is the dimensionless viscosity, c_F is the sound speed, r is the radial coordinate, H is the disk thickness, and all quantities are evaluated at the cooling front. The scaling exponent q lies in the interval $[0, 1]$, depending on the slope of the (T, Σ) relation in the hot state. For a Kramer's law opacity and $\alpha \propto (H/r)^n$, where n is of order unity, we find that $q \sim 1/2$. This supports the numerical work of Cannizzo, Chen and Livio (1995) and their conclusion that $n \approx 3/2$ is necessary to reproduce the exponential decay of luminosity in black hole X-ray binary systems. Our results are insensitive to the structure of the disk outside of the radius where rapid cooling sets in. In particular, the width of the rapid cooling zone is a consequence of the cooling front speed rather than its cause. We conclude that the exponential luminosity decay of cooling disks is probably compatible with the wave-driven dynamo model. It is not compatible with models with separate, constant values of α for the hot and cold states.

1. Introduction

The rate of mass transfer in accretion disks depends on the rate at which angular momentum can be transferred outward. This is normally expressed in terms of a dimensionless viscosity α, which is defined as

$$\alpha \equiv \frac{\nu}{c_s H}, \tag{1}$$

where H is the disk half-thickness, c_s is the local sound speed, and ν is the local effective viscosity (Shakura & Sunyaev 1973). Initially α was assumed to be constant. There are

now strong grounds, both empirical and theoretical, for concluding that α must be a variable. The task is to combine empirical evidence with theoretical guidance to construct a self-consistent theory of angular momentum transport in accretion disks that accounts for the wealth of observations. A successful theory is likely to reveal that angular momentum transport is non-local, so that the concept of a local viscosity is itself of limited value.

Communication between observational studies of accretion disks and theories of angular momentum transport is facilitated by models of time-dependent accretion disks. The time dependence is critical since the emissivity of steady-state disks is independent of the viscosity. An excellent review of the history of this area is given by Cannizzo (1993a; see also Cannizzo 1993b).

The first major constraint on α came from comparison of limit cycle disk instability models with observations of dwarf novae. The models provide a very credible basic interpretation of the dwarf nova phenomenon, but only if α is not a constant (Smak 1984). This constraint does not determine the functional form of α. Models in which α has one, radially constant value, α_{hot} in outburst and another lower, but also radially constant value α_{cold} in quiescence work about as well as a model in which $\alpha = \alpha_0(H/r)^n$ which would apply if α were a function of the sound speed and hence the temperature of the disk.

Another perspective on the behavior of α can be obtained by comparing the dwarf novae with soft X-ray transients. In the latter case, the only quantitative work has been done on those that are black hole candidates, but those are especially interesting laboratories because of the suspicion that the compact star lacks a hard surface and an associated magnetosphere and boundary layer. Several of the black hole candidates have outbursts with rapid rise and subsequent slower decline that are in reasonable agreement with the same limit cycle disk instability models that account for dwarf novae (Mineshige & Wheeler 1989).

The quantitative and even qualitative behavior of the black hole models depends on the prescription for α. In the case of a double valued, but radially constant prescription, the outburst will tend to occur in the inner disk, giving rise to somewhat slower rise, more symmetric outbursts. A prescription in which $\alpha = \alpha_0(H/r)^n$ will give very small values of α in quiescence where H/r is found to decrease inward. This will yield a very long viscous time in the inner disk and promote outbursts that begin in the outer disk and propagate inward. This yields model outbursts with rapid rise and slower decline, in accord with the observations for the optical and soft X-ray light curves of the X-ray novae.

One of the interesting features of the black hole X-ray novae is the tendency to show an exponential decline. Simple models in which one quickly reduces the transfer rate to a hot

disk with with constant α generate geometrically declining, not exponential, light curves. Even models in which the decline is driven by the cooling wave of the disk instability tend to have geometrically declining light curves with constant α. Mineshige et al. (1993) have argued that to produce an exponential decline, the angular momentum of the inner disk must be removed at a rate proportional to the angular momentum. They note that this tends to be the behavior of disk instability models with $\alpha = \alpha_0(H/r)^n$ with n $\sim$ 1 - 2. Cannizzo (1994) has also addressed this argument by noting that both dwarf novae and the black hole transients have exponential declines. Cannizzo concluded that to reproduce the exponential one needs $\alpha \propto r^\epsilon$, with $\epsilon \sim\ 0.3 - 0.4$, which is consistent with Mineshige et al. Cannizzo carried the argument one step further, however, by making the case that the precise value of ϵ that leads to exponential decline is itself a function of other parameters of the problem such as the transfer rate and inner disk radius. From this he concluded that exponential decline requires some form of feedback to operate in the disk to give just this behavior. This may hint that the angular momentum transport process is non-local, as the theories where internal waves play a critical role imply.

These arguments have been extended significantly by Cannizzo, Chen, and Livio (1995). Cannizzo et al. used well-resolved numerical studies to show that the width of the cooling front can be approximated very closely by w = $\sqrt{Hr}$ and that for such a behavior, exponential decay of the light curve during the cooling wave phase is obtained only for a prescription of the form $\alpha = \alpha_0(H/r)^n$ with n very close to 3/2. The critical point in their argument is that angular momentum is removed from the hot part of the disk by the advance of the cooling front and if the cooling front velocity is proportional to r, as it is for $n \sim 1.5$, then this loss of angular momentum is proportional to the total angular momentum in the disk. This preferred value of n is consistent with the theory of angular momentum transport by an internal wave-generated dynamo driven by tidal instabilities at the outer edge of the disk (Vishniac, Jin, & Diamond 1990, Vishniac & Diamond 1992). Nevertheless, the physical underpinnings of this behavior of the cooling wave were not clear. In particular, it is not clear why the cooling front should have this width. Given this width, it is possible to argue that the cooling front should have the velocity characteristic of torque induced mass flows with a radial length scale of w, i.e.

$$V_r \sim \frac{\alpha c_s^2}{w\Omega} \sim \alpha c_s \left(\frac{H}{r}\right)^{1/2}. \tag{2}$$

Since c_s near the cooling front is approximately constant, and since $H \sim c_s/\Omega \propto r^{3/2}$, this gives a cooling front velocity which is proportional to r when $n = 3/2$. In what follows we will argue that although this expression for the cooling front velocity is approximately correct, the direction of causality has been reversed. The cooling front width is a consequence of the cooling front velocity.

In this paper we present an analysis of the behavior of the cooling wave and show that its propagation depends on the viscous flow in the hot state and is nearly independent of the actual cooling process and of the state of the disk in the cool, quiescent material that accumulates in the wake of the inward-propagating cooling wave. We argue, in agreement with Cannizzo, Chen, and Livio, that the exponential decay gives strong evidence for the presence of the cooling wave, and hence of the disk instability phenomenon in general, and a powerful constraint on the physical nature of the local viscosity. The mechanisms that control the propagation of the cooling front are discussed in §2. Section 3 presents constraints on the opacity and other functional forms of α. The relation of these results to the internal wave-driven dynamo are presented in §4. Summary and conclusions are given in §5.

2. The Cooling Front

As the cooling front moves from large to small radii, it is preceded by a rarefaction wave that lowers the temperature and column density to the point where rapid cooling can set in. A rough sketch of the temperature and column density of the disk as a function of radius is shown in figure 1, with the sudden steepening of the temperature gradient to nearly a vertical line indicating the onset of rapid cooling. Figure 1 is merely illustrative of the cooling process, but is consistent with the detailed figures given by CCL. It is important to note that the radial distributions of column density and temperature of the disk are smooth power laws well inside the cooling front, but that they drop below an extrapolation of these power laws before the cooling front actually reaches them. We will refer to the region just inside of the cooling front where departures from the power law distributions occur as the precursor region.

We can understand the behavior of the cooling front by examining the equations for the conservation of mass and angular momentum, and the structure equations for a hot disk. These are

$$\partial_t \Sigma = -\frac{1}{r}\partial_r(r\Sigma V_r), \tag{3}$$

and

$$V_r = \frac{2}{\Sigma\Omega r^2}\partial_r\left(r^3\alpha\Sigma\frac{c_s^2}{\Omega}\partial_r\Omega\right), \tag{4}$$

where Σ is the gas column density, $\Omega(r)$ is the local rotational frequency (proportional to $r^{-3/2}$ in a Keplerian disk), α is the dimensionless viscosity, c_s is the local sound speed, and

V_r is the radial velocity. The thermal structure of an optically thick disk is determined by its opacity source. In general, the midplane temperature can be written as

$$T = B_1 \Sigma^{a_1} \alpha^{b_1} \Omega^{(2/3)c_1}. \quad (5)$$

In the hot portions of the disk for which the opacity can be approximated by a Kramers law opacity, $a_1 = c_1 = 3/7$ and $b_1 = 1/7$. Rapid cooling sets in for temperatures below T_{min}, where

$$T_{min} \propto \alpha^{-1.1/7} \Omega^{-\frac{3}{70}}, \quad (6)$$

(cf. CCL, equation (4)). It is important to note that T_{min} has a very weak dependence on radius.

In the hot inner part of the disk, well away from the cooling front, radial length scales are all comparable to r and equation (4) implies a radial velocity of order

$$|V_r| \sim \alpha \frac{c_s^2}{r\Omega}, \quad (7)$$

which is the usual result for a stationary accretion disk. In fact, the numerical simulations of CCL show that the inner disk is reasonably well described by the standard stationary disk solution. The mass transfer rate in the inner regions is changing with time, but is fairly constant with radius. Even though $\dot{M}$ reverses sign as one approaches the cooling front, the column densities and temperatures deviate appreciably from the stationary solution only close to the cooling front.

Before proceeding with a discussion of the cooling front physics, it will be useful to pause and consider the solutions to equations (3) and (4) in the hot part of a disk. Combining these expressions with equation (5), and assuming that $\alpha \propto (H/r)^n$, we can show that

$$2\pi r \Sigma V_r = \frac{\pi}{r\Omega} \left(-\frac{3}{2} \frac{k_B}{\mu} B_1 \right) \partial_r \left(r^2 \Sigma^{1+a_1} \alpha^{1+b_1} \Omega^{(2/3)c_1} \right), \quad (8)$$

or

$$\dot{M} = -C_2 r^{1/2} \partial_r (r^{\tilde{q}} \Sigma^{1/q}), \quad (9)$$

where C_2 is a constant,

$$\tilde{q} \equiv 1 + \left(1 + \frac{n}{2}\right) \left(\frac{1 - c_1}{1 - \frac{n}{2} b_1} \right), \quad (10)$$

and

$$q^{-1} \equiv 1 + \left(1 + \frac{n}{2}\right) \left(\frac{a_1}{1 - \frac{n}{2} b_1} \right). \quad (11)$$

If the disk is stationary then $\dot{M}$ will be a constant. If the inner, hot disk is slowly evolving, then $\dot{M}(t,r) \approx \dot{M}_0(t)$, and integration over r gives:

$$\dot{M}_0 \approx -\frac{C_2}{2} r^{\tilde{q}-\frac{1}{2}} \Sigma_0(r)^{1/q}, \tag{12}$$

or

$$\Sigma_0(r) \approx r^{(\frac{1}{2}-\tilde{q})q} \left(\frac{-2\dot{M}_0(t)}{C_2}\right)^q. \tag{13}$$

It is useful at this point to define a function $f(r) \equiv \Sigma/\Sigma_0$. If the hot, inner part of the disk is almost stationary then $f(r) \approx 1$ and $\partial_r f(r) \approx 0$ everywhere except close to the cooling front. Then we have

$$\partial_t \Sigma = \partial_t (f\Sigma_0) \approx f\partial_t \Sigma_0 = \frac{f}{q}\Sigma_0 \partial_t \ln(\dot{M}_0(t)). \tag{14}$$

Meanwhile, from equations (3), (9) and (12) we have

$$\partial_t \Sigma = -\frac{1}{2\pi r}\partial_r \dot{M} = -\frac{\dot{M}_0(t)}{2\pi r}\partial_r \left(f^{1/q}\left(1 + \frac{2}{q}\frac{\partial \ln f}{\partial \ln r}\right)\right). \tag{15}$$

If we take $f \approx 1$ then this can be simplified and combined with equation (14) to yield

$$\Sigma_0 q \partial_t \ln \dot{M}_0 \approx -\frac{\dot{M}_0(t)}{2\pi r}(\frac{3}{q}\partial_r f + \frac{2}{q} r \partial_r^2 f), \tag{16}$$

which has the solution

$$f \approx 1 - \left(\frac{r}{r_e(t)}\right)^m \tag{17}$$

where

$$m = 2 - q\tilde{q} + \frac{q}{2}. \tag{18}$$

Since Σ becomes small as r approaches $r_e(t)$ we can interpret $r_e(t)$ as a measure of the outer edge of the hot part of the disk. More generally, the actual edge of the hot phase of the disk will scale as $r_e(t)$, but will lie at some slightly smaller radius. Substituting this expression for $f(r)$ back into equation (16) we get

$$\Sigma_0 q \partial_t \ln \dot{M}_0(t) = \frac{\dot{M}_0(t)}{2\pi r^2}\frac{(2m^2+m)}{q}\left(\frac{r}{r_e(t)}\right)^m. \tag{19}$$

At this level of approximation we conclude that if we consider any fixed r well inside the radius of the cooling front we find that

$$r_e(t) \propto \left(-\partial_t \ln \dot{M}_0(t)\right)^{-1/m}\left[-\dot{M}_0\right]^{\frac{(1-q)}{m}}. \tag{20}$$

Consequently, if $r_e(t)$ is an exponentially decreasing function of time, then so is $\dot{M}_0(t)$. Inasmuch as the bolometric luminosity of the disk is determined by the mass accretion rate at small radii, this implies a direct connection between an exponentially decreasing radius for the hot portion of the disk, and an exponentially decreasing bolometric disk luminosity.

One other point is worth mentioning here. The radial velocity of the gas is (cf. eq (15))

$$V_r = \frac{\dot{M}}{2\pi r\Sigma} = \frac{\dot{M}_0(t)}{2\pi r\Sigma_0}(f^{1/q-1} + \frac{2}{q}rf^{1/q-2}\partial_r f) \approx \frac{\dot{M}_0(t)}{2\pi r\Sigma_0}f^{1/q-2}\left(1-(1+2\frac{m}{q})\left(\frac{r}{r_e}\right)^m\right). \quad (21)$$

This will have a zero when r is a fraction of $r_e(t) \approx 0.23$ for a Kramers-law opacity and $n = 1.5$. At this point $f \approx 0.8$, which is still close to unity. The simulations of CCL actually indicate that the radius of zero velocity will be at about 0.38 times the cooling front radius, but this is due to the fact that r_e is the radius at which the column density solution goes to zero, rather than the radius at which it drops below the hot phase minimum column density. The point at which $\mathrm{V}_R = 0$ is well defined in terms of the behavior of the velocity even though the disk temperature and column density scarcely depart from their steady-state values at that radius. Beyond this radius, the gas will move outward in the outer parts of the hot portion of the disk as the disk material is uniformly stretched. The column density will deviate from the stationary solution only close to the cooling front.

We could explore this expansion further, but since it becomes rapidly less accurate near the cooling transition, it seems unlikely to give us means of deriving the cooling front velocity. This does show that the mass flow rate at small r will decrease exponentially if the radius of the hot portion of the disk decreases exponentially *and if the cooling front moves slowly enough to allow the inner part of the disk to stay close to a stationary solution.* We will see later that this condition is automatically satisfied.

In order to understand the cooling front velocity we need to focus on the structure of the disk near the cooling front. Let us consider some fiducial point just inside of the precursor region and denote quantities at that radius with a subscripted p. Quantities evaluated at the cooling front will be denoted with a subscripted F. Mass enters the precursor at a rate $\Sigma_p(V_r(r_p) + v_{cF})$, where v_{cF} is the cooling front velocity. This mass flow should be balanced, allowing for some slight difference in r_p and r_F and the secular evolution of the cooling front, by the flow of material into the region of rapid cooling. At the onset of rapid cooling the gas begins to cool at some large fraction of the thermal rate, which for an optically thick disk is $\sim \alpha\Omega$. If the radial scale for temperature change is L_F then from equation (4) we have

$$V_F \sim \alpha_F \frac{c_F^2}{L_F\Omega_F}, \quad (22)$$

where V_F, the radial velocity at the cooling front, is positive since the thermal gradient

is strongly negative. Inasmuch as the material is essentially freely cooling at this point $L_F \sim V_F/(\alpha_F \Omega_F)$ which implies

$$V_F \sim \alpha_F c_F. \tag{23}$$

Detailed comparison with CCL's results shows that the radial velocity at the cooling front is actually only about $(1/6)\alpha_F c_F$. On the other hand, this fraction is constant in time and our main concern here is with the scaling properties of the cooling front rather than with a derivation of the various numerical factors which figure in the actual solution. Matter conservation implies that

$$\Sigma_p(V_r(r_p) + v_{cF}) \sim \Sigma_F \alpha_F c_F. \tag{24}$$

Here we have assumed that the mass velocity at the cooling front, V_F, is much larger than the cooling front velocity, v_{cF}.

We can simplify equation (24) further by arguing that v_{cF} cannot be arbitrarily smaller than $V_r(r_p)$. If it were, then the disk would evolve ahead of the cooling front faster than the cooling front could propagate. In particular, the outer edge of the hot portion of the disk, just ahead of the cooling front, would become depleted of all matter before the cooling front reached it. Since equations (5) and (6) imply the existence of a minimal column density, below which rapid cooling sets in, this situation is paradoxical. A minimal speed for the cooling front is supplied by the rate at which the outer edge of the hot portion of the disk would move inward purely from the depletion of matter, i.e. the viscous accretion speed. We conclude that v_{cF} is either scaling with $V_r(r_p)$ or becomes increasingly larger than it as r_F decreases. Therefore equation (24) can be rewritten as

$$v_{cF} \sim \frac{\Sigma_F}{\Sigma_p} \alpha_F c_F. \tag{25}$$

It is also evident that the cooling front cannot move arbitrarily faster than $V_r(r_p)$. If it did, then the inner disk would be unable to evolve before the cooling front reached it. Consequently, Σ_p would be virtually unchanged from its value at the epoch when the cooling front first formed at the outer edge of the disk. From equation (4) and (5) it is straightforward to show that for a Kramers law opacity and α constant

$$\Sigma \propto r^{-0.75}, \tag{26}$$

in a stationary disk. Under the same assumptions, equations (5) and (6) imply that Σ_F can increase almost as fast as r (for a constant α). Equation (25) then implies that v_{cF} scales roughly as $r^{(7/4)}\alpha_F$, or $r^{(7/4)+n/2}$. This steep positive scaling implies that v_{cF} will drop rapidly as the cooling front moves inward. Substituting functional forms of α that are consistent with disk observations changes the values of the exponents slightly, but leads to

the same qualitative conclusion. On the other hand, the condition that the mass flow rate, $\dot{M} = rV_r\Sigma$, is a constant implies that V_r *rises* slowly as $r \to 0$. We conclude that if the ratio of v_{cF} to $V_r(r_p)$ is large then it will decrease rapidly with decreasing r.

Since neither a large nor small ratio of v_{cF} to $V_r(r_p)$ is sustainable we conclude that the cooling front will evolve into a state where the two scale together, i.e.

$$v_{cF} \sim V_r(r_p) \approx \alpha_p \frac{c_p^2}{r_p\Omega(r_p)}. \tag{27}$$

We could have shortened the derivation of equation (27) somewhat if we had simply defined the fiducial point p to lie at the radius where the fluid velocity vanishes. Then the mass flow into the precursor region would have been v_{cF} and $V_{r_p} = 0$ by definition. While this would have eliminated most the argument preceding equation (25), it would meant that the factors of (r_p/r_F) in our scaling relations could not be assumed to be ≈ 1. In particular, we would have been forced to assume that such factors were approximately constant, which is consistent with the numerical simulations, but not otherwise justified in this paper. It is more convenient to take p close to the cooling front, although with a radial scale length still of order r, and consequently a radial velocity of the order given in equation (27).

Equations (5), (6), (25), and (27) are all we need to solve for the scaling properties of v_{cF} for any given functional form of α. Combining equations (25) and (27) we have

$$\alpha_p \frac{c_p^2}{r_p\Omega(r_p)} \sim \frac{\Sigma_F}{\Sigma_p}\alpha_F c_F. \tag{28}$$

Since the precursor front is narrow we can replace r_p with r_F. (Actually, since we are only concerned with scaling laws, this argument would work for a broad precursor as long as r_p were some fixed fraction of r_F.) Rearranging terms we find

$$\frac{\alpha_p}{\alpha_F}\frac{T_p\Sigma_p}{T_F\Sigma_F} \sim \frac{r_F\Omega_F}{c_F}. \tag{29}$$

In order to proceed beyond this point it is necessary to choose some form for α. We start with the form $\alpha = \alpha_0(H/r)^n$, which was used by CCL and is motivated by the various theoretical and phenomenological arguments cited in the introduction. This implies $\alpha \propto (rT)^{n/2}$. Consequently, equation (5) becomes

$$T^{1-b_1\frac{n}{2}} \propto \Sigma^{a_1} r^{b_1\frac{n}{2}-c_1}, \tag{30}$$

and equation (29) becomes

$$\left(\frac{T_p}{T_F}\right)^{1+\frac{n}{2}} \frac{\Sigma_p}{\Sigma_F} \sim \frac{r_F\Omega_F}{c_F}. \tag{31}$$

Combining these two equations yields

$$\frac{\Sigma_F}{\Sigma_p} \sim \left(\frac{c_F}{r_F \Omega_F}\right)^q, \tag{32}$$

or,

$$v_{cF} \sim \alpha_F c_F \left(\frac{c_F}{r_F \Omega(r_F)}\right)^q \sim c_F^{q+n+1} r_F^{\frac{q+n}{2}}, \tag{33}$$

where q is defined in equation (11).

CCL suggested $q = 0.5$ on the basis of their numerical simulations. Here we see that its value is a function of the opacity law in the hot state and the value of n. For a Kramers law opacity we get $q = 0.54$ for $n = 3/2$ with q dropping to 0.5 for $n = 2$ and rising to 0.59 for $n = 1$. In spite of its functional dependence on n it is difficult to get q very different from 0.5 for any reasonable choice of n. The result is similarly insensitive to the exact opacity law. If we consider electron scattering instead, for which $a_1 = 2/3$, $b_1 = 1/3$, and $c_1 = 1/2$, we find that $q = 0.39$ for $n = 3/2$. Finally, we note that this argument does not include the dynamics of the rapid cooling region, or the cold state, at all. These can be varied in any way that preserves the existence of a rapid cooling zone without changing the cooling front speed.

In their paper CCL proposed that n should be close to 3/2, since that was the value that gave an acceptably exponential decline in the disk luminosity as the cooling front propagated inward. More specifically $v_{cF} \propto r_F$ implies an exponential decline, so treating T_F (which is also T_{min}; eq (6)) and hence c_F as approximately constant and $q = 0.5$ implies $n = 3/2$, as can be seen from eq (33). Strictly speaking, numerical models show that T_{min} depends somewhat on radius. Given our form for α we can rewrite equation (6) as

$$T_F \propto r^{(\frac{9}{140} - \frac{1.1}{7}\frac{n}{2})/(1+\frac{1.1}{7}\frac{n}{2})}. \tag{34}$$

Substituting this into equation (33) we see that

$$v_{cF} \propto r^{0.949}, \tag{35}$$

for $n = 1.5$. and

$$v_{cF} \propto r, \tag{36}$$

when $n = 1.65$. We see that our results suggest that slightly higher values of n are necessary to obtain a purely exponential decay when the temperature at which cooling sets in depends somewhat on radius. Of course, the observations themselves suggest only approximately exponential decay. The question here is whether or not the cooling front dynamics are actually controlled by the preceding rarefaction wave as our model assumes. More recent calculations (Cannizzo 1996) show that $n = 1.625$ does indeed produce a more nearly exponential form than $n = 1.5$.

3. Alternative Opacities and Functional Forms For α

We have seen that the link between the observed exponential decay of soft X-ray luminosity of black hole binary systems and the conclusion that $\alpha \propto (H/r)^{3/2}$ depends on the opacity law for the hot state of the disk, as well as our initial assumption of a functional form for α. In this section we will examine the consequences of taking other opacity laws and ask whether or not there are other forms for α that would do as well. We will defer discussion of a nonlocal model for α, the internal wave driven dynamo, until the next section.

It is helpful to begin by assuming that the hot disk is optically thick, with an opacity law of the form

$$\kappa \propto \rho^A T^{-B} \tag{37}$$

so that the disk opacity is

$$\tau \sim \kappa\Sigma \propto \Sigma^{A+1} T^{-B} H^{-A} \propto \Sigma^{A+1} T^{-B-\frac{A}{2}} \Omega^A. \tag{38}$$

Invoking the equality between the energy generation rate per unit area, $\dot{M}\Omega^2 \sim \alpha\Sigma c_s^2\Omega$, and the rate at which energy is radiated, $\sigma_B T^4/\tau$, we can recover an equilibrium relationship of the form given in equation (5) with

$$a_1 = \frac{A+2}{3+B+A/2}, \tag{39}$$

$$b_1 = \frac{1}{3+B+A/2}, \tag{40}$$

and

$$c_1 = \left(\frac{3}{2}\right) \frac{A+1}{3+B+A/2}. \tag{41}$$

Limits on the plausible range of these parameters, and more importantly for the exponent q in equation (11), can be deduced from the requirement that the hot phase of the disk must be thermally and viscously stable. Thermal stability implies that for a fixed Σ and r, a small positive deviation of T away from equilibrium will produce a larger increase in the cooling rate than in the heating rate. The cooling rate per unit area is proportional to $T^4\tau^{-1}$ or

$$Q^- \propto T^4\tau^{-1} \propto T^{4+B+\frac{A}{2}}. \tag{42}$$

The heating rate per unit area is

$$Q^+ \propto \dot{M} \propto \alpha T. \tag{43}$$

If we use

$$\epsilon \equiv \frac{\partial \ln \alpha}{\partial \ln T}, \tag{44}$$

to parameterize the dependence of α on temperature, then the hot phase will be thermally stable if

$$1 + \epsilon < 4 + B + \frac{A}{2}. \tag{45}$$

Remembering that for $\alpha \propto (H/r)^n$ we have $\epsilon = n/2$, we see that equation (11) implies that $q \to 0$ as the hot phase approaches the threshold of thermal instability. In other words, as we consider opacities for the hot phase that bring it closer and closer to a loss of thermal stability, the cooling front moves closer and closer to its maximum speed of $\alpha_F c_F$.

The condition that the hot phase is viscously unstable is that

$$\frac{\partial \dot{M}}{\partial \Sigma} < 0, \tag{46}$$

where thermal stability is assumed. This can be rewritten as

$$1 + \frac{\partial \ln \alpha}{\partial \ln \Sigma} + \frac{\partial \ln T}{\partial \ln \Sigma} > 0, \tag{47}$$

or

$$\frac{\partial \ln T}{\partial \ln \Sigma}\left(1 + \frac{\partial \ln \alpha}{\partial \ln T}\right) > -1. \tag{48}$$

We can see by comparing this result to equations (5) and (11) that viscous instability will set in only when q passes through ∞ to negative numbers. In other words, an arbitrarily slow cooling front can be produced by letting the hot phase go to the threshold of viscous instability.

It seems odd that it is possible to get a cooling front moving more slowly than the viscous accretion speed. The reason this works is that when a (as defined in equation (5)) is negative, the rarefaction wave that precedes the cooling front actually raises the temperature of the gas, making it more difficult for it to reach the minimum hot phase temperature. In fact, it is difficult to see how we can get a cooling front when this is the case. If we adopt the more reasonable requirement that T increase with Σ (i.e. $a > 0$) then the slowest possible cooling front speed is just $\alpha_F c_F (H_F/r_F)$, when $a = 0$ (and $q = 1$). In other words, as the midplane temperature becomes insensitive to the column density, the cooling front speed converges to the accretion velocity in the hot state. It seems that for any reasonable opacity law for the hot state, the exponent q will fall in the interval $[0, 1]$. Since the limits describe fairly extreme situations, we expect that typically q will fall somewhere near the middle of this interval, even when the hot state does not have a Kramer's law opacity.

Now we consider the effect of using a different functional form for α. The behavior of the cooling front depends only on the properties of the hot state, including the functional form of α. This implies that a model where α is given by a pair of values, i.e. $\alpha = [\alpha_{hot}, \alpha_{cold}]$ is equivalent to taking $n = 0$ in the preceding section. In this case the cooling front speed is

$$v_{cF} \sim \alpha_{hot} c_F \left(\frac{c_F}{r_F \Omega(r_F)} \right)^{\frac{1}{1+a}} \propto c_F^{1.7} r^{0.35}, \tag{49}$$

which is unacceptably different from $v_{cF} \propto r$.

Alternatively, one could take $\alpha \propto r^k$. This form has the problem that α does not increase as one passes from the cold state to the hot state, and for that reason is unlikely to produce a fit to the entire outburst cycle. However, we can ask the narrower question of whether or not such a form could produce an exponential decay. Since α is a function of r only, and since r_p and r_F differ by only a constant factor (which will be close to one in most cases) equation (5) implies that

$$\frac{T_p}{T_F} = \left(\frac{\Sigma_p}{\Sigma_F} \right)^{3/7}. \tag{50}$$

Then equation (29) implies

$$\left(\frac{\Sigma_p}{\Sigma_F} \right)^{10/7} \sim \frac{r_F \Omega_F}{c_F}. \tag{51}$$

Combining this result with equations (25) and (6) we find that

$$v_{cF} \sim \alpha(r_F) c_F \left(\frac{c_F}{r_F \Omega(r_F)} \right)^{0.7} \propto r^{0.405+0.866k}. \tag{52}$$

We conclude that for $\alpha \propto r^k$ an exponential decay law will follow from $k \approx 0.69$.

4. The Internal Wave Driven Dynamo Model

The internal wave driven model (Vishniac, Jin, & Diamond 1990, Vishniac & Diamond 1992, and Vishniac & Diamond 1993) predicts that α has the form assumed in CCL, with $n \approx 3/2$. (The precise value depends somewhat on the way the wave cascade is modeled, with n as low as $4/3$ possible if every level in the cascade contributes maximally to the dynamo process.) While it is exciting to see this prediction validated by CCL, the internal wave driven dynamo model is inherently nonlocal, and it is unclear whether or not the arguments given in the preceding section can be applied to it. More specifically, in this

model the local dynamo activity, and the consequent turbulent viscosity, depend on the amplitude of the internal wave field. These waves propagate inward and are excited by tidal instabilities near the outer edge of the disk (Goodman 1993, Ryu & Goodman 1994, Ryu, Goodman, & Vishniac 1996, Vishniac & Zhang 1996). Normally their amplitude can be estimated by balancing nonlinear dissipation with their linear amplification and focusing as they propagate inward. This implies a mean square wave amplitude proportional to (H/r). However, near a cooling front the waves travel across a region of relatively sharp increase in this ratio and one would expect that the wave amplitudes would fall well below their usual saturation value. It follows that in order to test the consistency of the internal wave driven dynamo model there are several issues that must be addressed. First, what is the effect of decreasing α_F below the value inferred from the equilibrium expression? Second, how does the amplitude of the internal waves change as they cross the cooling front? Third, how does this decrease translate into a reduced value for α_F?

We could add to these the question of whether or not the amplitude of the α inferred from the luminosity decay in X-ray binaries is consistent with this model. Unfortunately, the model is not yet sophisticated enough to predict this constant, so this test must be deferred until a later time.

We will assume from the start that away from the cooling front $\alpha = \alpha_0 (H/r)^n$ and the hot state opacity is given by Kramers law. Then we define the softening factor f by

$$\alpha_F = \alpha_0 f \left(\frac{H}{r}\right)^n. \tag{53}$$

In this case equation (29) implies

$$\frac{r_F \Omega_F}{c_F} \sim \frac{\alpha_p}{\alpha_F} \frac{T_p \Sigma_p}{T_F \Sigma_F} = f^{-1} \left(\frac{T_p}{T_F}\right)^{1+n/2} \frac{\Sigma_p}{\Sigma_F}. \tag{54}$$

From equation (5) we have

$$\left(\frac{T_p}{T_F}\right)^{1-\frac{n}{14}} = f^{-\frac{1}{7}} \left(\frac{\Sigma_p}{\Sigma_F}\right)^{3/7}, \tag{55}$$

for a Kramer's law opacity. Combining this with equation (54) we get

$$\frac{r_F \Omega_F}{c_F} \sim f^{-1-\frac{1+n/2}{7-n/2}} \left(\frac{\Sigma_p}{\Sigma_F}\right)^{1/q}, \tag{56}$$

where q is given in equation (11). This gives us the dependence of the ratio of column densities on f. The cooling front velocity is also modified by the rise in T_{min} (and therefore c_F) caused by the drop in α_F. Combining equations (6), (25) and (56) we conclude that

$$v_{cF} = v_{cF0} f^k, \tag{57}$$

where

$$k = \frac{9.3\left(1+\frac{n}{2}\right)}{(10+n)(14+1.1n)}, \tag{58}$$

and v_{cF0} is the cooling front velocity when $f = 1$. The exponent k is small and relatively insensitive to n. For n close to 1.5, $k \approx 0.09$. It seems clear that f has to vary strongly with radius in order to change the scaling of the cooling front speed with radius.

A more significant problem is that for f very small, the characteristic S-shape equilibrium curve in the $\Sigma - T$ plane disappears, the cooling transition becomes gradual, and the front speed dynamics become more complicated. When this happens will depend critically on the equilibrium curve for the cold state, which has not previous affected our calculations. We will not discuss this question further, but note that it should be a focus of subsequent work.

In order to decide whether or not the small change in the scaling relationship induced by f will create problems for the internal wave driven dynamo model, we need to examine the scaling of α in this model with the wave amplitude. For a review of this model see Vishniac & Diamond (1993). We will quote selected results here. The internal waves responsible for driving the dynamo consist of slightly non-axisymmetric ($m = 1$) waves whose amplitude is determined by the balance between linear amplification, which occurs at a rate of roughly

$$\tau_{amp}^{-1} \sim \frac{V_{group} m}{r} \sim \left(\frac{H}{r}\right)\Omega, \tag{59}$$

and nonlinear damping, which occurs at a rate of roughly

$$\tau_{nonlinear}^{-1} \sim \mathcal{M}^2\left(\frac{\Omega^2}{\bar{\omega}}\right). \tag{60}$$

Here $\bar{\omega}$ is the comoving frequency of the wave, V_{group} is the radial group velocity of the waves, and $\mathcal{M}$ is their Mach number. For these waves

$$V_{group} \sim \left(\frac{\bar{\omega}}{\Omega}\right)^2 c_s. \tag{61}$$

Nonlinear interactions will keep $\bar{\omega}/\Omega$ of order unity as long as the underlying disk changes only on length scales of order r. This implies that normally $\mathcal{M}^2 \sim (H/r)$. However, as the waves pass through the cooling front their group velocity increases. Conservation of wave energy flux implies

$$\left(\frac{\mathcal{M}_F}{\mathcal{M}_{cold}}\right)^2 = \frac{\Sigma_{cold}}{\Sigma_F}\left(\frac{T_{cold}}{T_F}\right)^{3/2}. \tag{62}$$

We will define this ratio as the factor D. Since the waves start out in the cold state with an amplitude that roughly balances linear amplification with nonlinear dissipation, and

since the nonlinear dissipation rate is proportional to $\mathcal{M}^2$, D is also the ratio of nonlinear dissipation rate to the linear amplification rate just inside the cooling front. In CCL the ratio of column densities was of order 10, while the temperature ratio was of order 10^2, implying $D \sim 10^{-2}$. However, this temperature ratio results from extrapolating using a power-law fit to the cold state models with midplane temperatures of a few thousand degrees. While the cold state opacities are not completely known, the actual midplane temperatures for the cold state probably fall in the range $1 - 3 \times 10^3$ K. This implies a temperature ratio of somewhat more than 10 across the cooling front. On the other hand, the column density ratio should be of order $\frac{\Sigma_p}{\Sigma_F}$, since the material moves rapidly only near the cooling front itself and will have a velocity, relative to the cooling front, of order v_{cF} both in the the cold state just outside of the cooling front and just inside the precursor region. Consequently, this ratio will be largely unaffected by uncertainties in the cold state $\Sigma - T$ relation. It seems safe to conclude from this that D is generally of order 10^{-1}.

How can we obtain f from D? This turns out to hinge on which waves are responsible for driving the dynamo process. For an 'α' driven by magnetic turbulence the value of α is roughly the dynamo growth rate divided by Ω. For the wave-driven dynamo this is

$$\alpha \sim \frac{\Gamma_{dynamo}}{\Omega} \sim \left(\mathcal{M}^2 \frac{m}{\bar{\omega}\tau_{nonlinear}} \left(\frac{H}{r}\right) \Delta\right)^{1/2}. \tag{63}$$

Here Δ is the fractional asymmetry of the wave field. Only modes with the same sign of mk_r add coherently. If we restrict ourselves to the $m = 1$ modes (cf. Vishniac, Jin & Diamond 1990) then this implies $\alpha = \alpha_0 (H/r)^{3/2}$, where the constant α_0 is undetermined. However, the process by which the internal waves dissipate is through a period doubling cascade, which is eventually truncated through interactions with the magnetically driven turbulence. The dissipation rate of the waves by the turbulence goes as $\alpha(\Omega^3/\bar{\omega}^2)$ and the end point of the cascade is obtained by equating this to $\tau_{nonlinear}^{-1}$. If we assume that $\Delta \approx 1$ throughout the cascade (cf. Vishniac & Diamond 1992) then this implies $\alpha = \alpha_0 (H/r)^{4/3}$.

The difference of a factor of $(H/r)^{1/6}$ has not previously been important, but it may be in this context. In addition, for the purpose of determining $f(D)$ it matters a great deal whether or not the dynamo process is controlled by the waves at the top or the bottom of the cascade. Conservation of radial and azimuthal momentum in the cascade turns out to imply that Δ has to decrease at least as fast as $\bar{\omega}/\Omega$ as one goes to smaller $\bar{\omega}$, which rules out $\alpha \propto (H/r)^{4/3}$. However there are limits to how quickly Δ can decrease with $\bar{\omega}$. For any wave $\bar{\omega}$ is a function of position, and an asymmetry in the wave field is created by normal wave propagation. The size of this effect is roughly

$$\Delta \sim V_{group} \tau_{nonlinear} \partial_r \ln \bar{\omega} \sim \bar{\omega} \tau_{nonlinear} \frac{mH}{r}. \tag{64}$$

Combining this result with equation (63) gives

$$\alpha \sim \frac{H}{r}\mathcal{M}|m|. \tag{65}$$

Conservation of wave energy in the cascade implies that $\mathcal{M}^2 \propto \bar{\omega}^{1/2}$. If m increases as we go down the cascade though a random walk process then $m \propto \bar{\omega}^{-1/2}$ and invoking the condition for the turbulent truncation of the cascade gives

$$\alpha = \alpha_0 \left(\frac{H}{r}\right)^{10/7}, \tag{66}$$

when $D = 1$. In other words, although the cascade does dominate the total helicity, and the subsequent value of α, it increases the dynamo growth rate by only a small factor. When $D < 1$ the cascade will extend to lower $\bar{\omega}$. In addition, the upper end of the cascade will change slightly, since for small D the linear evolution of the waves is much more important than nonlinear dissipation. For simplicity we will ignore such effects here. Ignoring the entire cascade and considering only the fundamental modes gives $\alpha \propto D$. Attempts to model the effect of $D < 1$ on the whole cascade change this very slightly. This implies that the internal wave driven dynamo contribution to α drops by more than an order of magnitude across the cooling front. The exact amount and its scaling with radius will depend on details of the cold state opacity.

Fortunately, there is a simpler solution to this problem. The turbulence induced by magnetic field instabilities can support a dynamo in a shearing environment even in the absence of any long term average helicity (Vishniac & Brandenburg 1996). For an accretion disk this gives $\alpha \propto (H/r)^2$. The constant of proportionality is different from α_0 and is again unknown. Lacking an accurate estimate we will assume that it is comparable to α_0. Since $(H/r)^{1/2}$ is only slightly less than 10^{-1} for these disks, it seems reasonable to take $f \propto (H/r)^{1/2}$. From equations (57), (58), and (66) we see that this implies the cooling wave moves at the speed one would expect for $n = 10/7 + 0.045 \approx 1.47$.

Of course, if f is small enough the dynamics of the cooling front may be significantly altered. Further work on this, including a derivation of the scaling coefficients of the internal wave driven dynamo and the incoherent dynamo, are necessary to fully answer the question of whether or not the internal wave driven dynamo is consistent with observations of post-outburst luminosity decline in accretion disks.

5. Summary and Conclusions

We have constructed a simple model for the propagation of cooling fronts in accretion disks which reproduces the numerical results of Cannizzo, Chen, & Livio (1995). In this model the cooling front speed is determined by the rarefaction wave that lowers the disk temperature to the point where rapid cooling can set in. We find that the speed of the cooling front scales as

$$v_{cF} \sim \alpha_F c_F \left(\frac{c_F}{r_F \Omega(r_F)}\right)^q, \tag{67}$$

where the subscript F refers to the radius where the disk falls out of thermal equilibrium and begins rapid cooling. The coefficient q is given in equation (11) and depends on both the opacity law and the functional form of α. However, q will be close to $1/2$ for most models of the disk hot state. Somewhat surprisingly, a local reduction of α near the cooling front has only a modest effect on this result.

One striking aspect of our derivation is that we do not appeal to any aspect of the structure of the disk at radii greater than r_F, which marks the onset of rapid cooling. This stands in contrast to the suggestion by CCL that the cooling front velocity is determined by its width, measured from the onset of rapid cooling to its finish. We have not discussed the structure of this region here, but we note that given a cooling front velocity determined by the structure of the disk inward from the cooling front, the width of the cooling front itself can be estimated by inverting equation (2). In other words, the scaling of the cooling front width is a consequence of the speed of the cooling front, not its cause.

Our success in modeling the propagation of cooling fronts as rarefaction waves suggests a similar effort could be made to model heating fronts as compressional waves. We have not yet done this, but expect to examine this problem in a future paper.

Aside from the internal wave driven dynamo model, we have not discussed models for α which are consistent with the results of this paper. While this is largely from a lack of suitable candidates, there is one other prediction of α with the required form (Meyer & Meyer-Hofmeister 1983). However, this estimate is based on using large scale buoyant cells driven by magnetic buoyancy via the Parker instability (cf. Parker 1975). Zweibel & Kulsrud (1975) showed that sufficiently strong turbulence in a shearing environment would suppress the Parker instability. Vishniac & Diamond (1992) pointed out that the Balbus-Hawley instability (Velikhov 1959, Chandrasekhar 1961, Balbus & Hawley 1991) always leads to a level of turbulence which is sufficiently strong by this criterion. In fact, the linearly unstable modes of an azimuthal magnetic field suffer turbulent mixing at a rate roughly equal to the local orbital frequency. We conclude that the magnetic buoyancy driven model is not consistent, in its original form, with the dynamics of magnetic fields in accretion disks.

There are several conclusions to be drawn from this work.

First, we have provided a simple analytic derivation which supports the conclusion of CCL that the exponential decay of the luminosity of black hole disk systems following outbursts is consistent with a local law for the dimensionless disk viscosity $\alpha \propto (H/r)^n$ if, and only if, n is approximately $3/2$.

Second, given this scaling for α we find that disk systems in general should exhibit approximately exponential luminosity decay from peak luminosity whenever the hot state opacity follows a simple power law. Exceptions will involve hot state (Σ, T) relations which are either unusually close to thermal instability, in which case the cooling front velocity can approach $\alpha_F c_F$, or in which T is extremely insensitive to Σ, in which case the cooling front velocity will approach the accretion velocity in the inner disk.

Third, this result implies that any α scaling for which α is constant in the hot state is in conflict with current observations. This includes models in which α is given by $[\alpha_{hot}, \alpha_{cold}]$, where α_{hot} is a constant.

Fourth, since the cooling front speed depends only on the hot state, other models for α can also give exponential decays, although they may fail on other grounds. For example, if $\alpha \propto r^{2/3}$, then we can obtain a roughly exponential luminosity decay in spite of the fact that this law is insensitive to the local temperature.

Fifth, this result is apparently compatible with the internal wave driven dynamo model for disk viscosity. This does not follow trivially from the prediction that $\alpha \propto (H/r)^n$, where n is approximately $3/2$ in a stationary disk. The waves reach the cooling front after traveling through the cold part of the disk. Consequently, the α_F induced by the internal wave driven dynamo is greatly reduced. Here we have relied on an independent mechanism, the incoherent dynamo, to give a minimal value for α_F. The final scaling law obtained in this way lies within observational limits. Clearly further work on these dynamo mechanisms, and on the nature of the cold disk state, would be helpful for providing a definitive answer to this question. Still, this is the only internally consistent model for α which is constructed from first principles and which satisfies the cooling wave constraint. We note that CCL have shown that the value of α_0 can be estimated from the luminosity decay rate. This value has not been calculated for the internal wave driven dynamo model, but when it is the existence of an observationally motivated estimate will provide another critical test of the model.

Finally we note that this whole analysis is predicated on the assumption that a cooling wave exists in the decline of the light curve of transient black hole candidates and related systems. While the evidence is indirect, one can thus regard the exponential decline as

a strong argument that a cooling wave is the fundamental mechanism of the decline of these transients. Furthermore, it adds to the evidence that the accretion disk ionization instability is the underlying physical cause of the transient outburst phenomenon.

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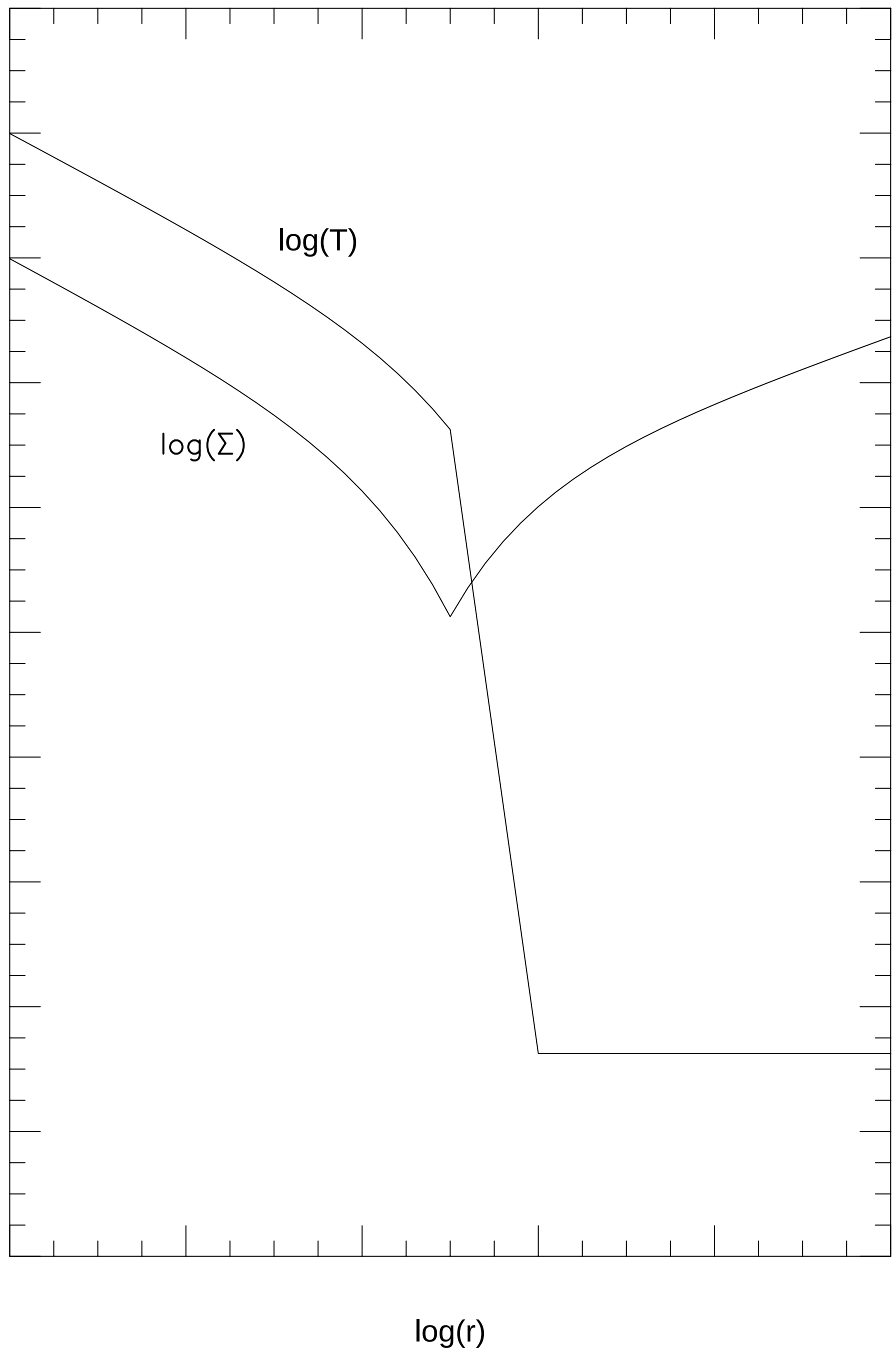

Fig. 1.— A schematic of the variation of temperature T and column density Σ as a function of radius r across a cooling front propagating towards small r.